Phenomenological Model for Dispersed Bubbly Flow in Pipes

Dimitris E. Nikitopoulos
Dept. of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803

Efstathios E. Michaelides
Dept. of Mechanical Engineering, Tulane University, New Orleans, LA 70118

An analytical approach to the problem of steady-state, axisymmetrically dispersed, bubbly flow in pipes based on a zero equation turbulence model is discussed. The formulation incorporates recent experimental observations and introduces the effect of bubble size in a rudimentary way. The two-phase mixture is modeled as a variable-density single fluid assuming an empirical void distribution family. The turbulent shear stress is formed from the contributions of both the velocity and density variation, and the solution of the resulting Reynolds-type equation yields the velocity profile of the flow. Predicted void fraction and velocity distributions agree well with experimental measurements. The main objective of the model is to predict the friction multiplier with minimal computational effort. The velocity profiles of this model agree reasonably well with experiments. Predictions for the friction multiplier are compared to six known and widely used correlations, as well as to experimental data. All the correlations severely underpredict the friction multiplier in the dispersed bubbly flow regime, while the results of our model agree well with the measurements, within the range of its validity.

Introduction

The ability to predict pressure drop as a function of flow parameters for gas-liquid pipe flows in general and dispersed, bubbly, pipe flows in particular is of considerable importance to a number of industries that utilize two-phase flow systems and processes (for instance, petroleum, chemical, nuclear, and geothermal). The great significance of pressure drop prediction is reflected in the large numbers of models and correlations that have been introduced over the years. The fully empirical approach to the problem through correlations derived from experimental data banks is dominant in the existing literature. Pressure drop correlations such as the ones by Lockhart and Martinelli (1949), Armand (1950), Baroczy (1966), Orkiszewski (1967), Chisholm (1973), Dukler et al. (1964), Friedel (1979), and Beattie and Whalley (1982) are representative of the kind and have found wide use in practice. Empirical correlations, however, lack a firm physical basis and cover a limited range of physical parameters. Consequently, predictions of the frictional pressure drop resulting from correlations can often be off by as much as an order of magnitude. This was clearly shown by Nakoryakov et al. (1981) in the case of low void fraction, dispersed, bubbly pipe flow, from their comparison of the measured frictional pressure drop with that predicted by the correlation of Armand (1950), which provides values that are not much different from the other correlations listed above. This discrepancy between correlation predictions and Nakoryakov's data was also pointed out by Clark and Flemmer (1985). Recognizing the weaknesses of the fully empirical approach, many investigators have directed their efforts toward a more fundamental one based on physical principles and conservation laws.

A simple analytical approach to the subject is the single fluid one, first introduced by Bankoff (1960), where the two-phase mixture is considered as a single fluid with variable density. Bankoff assumed power law distributions for the velocity and void fraction in the "bubble" regime and proceeded to derive expressions for the frictional pressure drop and the slip between the phases. Zuber and Findlay (1965) improved on Bankoff's approach by introducing relative velocity effects through their drift flux model. Levy (1963) and later Maeder and Michaelides

Correspondence concerning this article should be addressed to D. E. Nikitopoulos.

(1980) used a solution of the momentum equation and incorporated Prandtl's mixing length ideas into their variable density fluid model with a high degree of success. More recently, Clark and Flemmer (1985a,b) introduced the concept of excess shear caused by the local increase of the fluid velocity due to the buoyancy-driven, faster-moving bubbles. Their model, which uses mixing length theory and a postulated power law void distribution, involves two adjustable constants in the expression for the wall shear stress which is given as a function of average void fraction and liquid superficial velocity. For an appropriate combination of the two constants, the model of Clark and Flemmer (1985b) predicts the frictional pressure drop, measured by Nakoryakov et al. (1981) at low volumetric gas-flow ratios, reasonably well. The mixing length approach has also found wide application in bubble columns where the average velocity of the carrier liquid phase is zero. Since our interest here lies in pipe flows where the mean liquid velocity is nonzero, the body of literature on bubble columns will not be discussed.

Several studies have been undertaken to examine how fluid turbulence and bubble distribution influence each other. Theofanous and Sullivan (1982) examined theoretically and, to a lesser extent, experimentally the enhancement of liquid turbulence by flowing bubbles, and Drew and Lahey (1982) developed an analytical model that qualitatively predicted void fraction peaking off the center of the pipe for upward flows. Malnes (1966) obtained phase and velocity distribution data, while Serizawa et al. (1975) and Serizawa and Michiyoshi (1984) obtained experimental results for the influence of bubbles on turbulence and the effect of the latter on the velocity and void distributions. Nakoryakov et al. (1981) made extensive measurements of wall shear stress (the only ones available in dispersed bubbly pipe-flow) as well as void fraction and velocity distribution in dispersed, bubbly pipe flow. Phase distribution and velocity measurements have also been recently reported by Wang et al. (1987) and Liu and Bankoff (1990). The latter study is the only one to provide bubble chord-length distribution data.

It should be noted that all of the above-mentioned experimental studies, except for Theofanous and Sullivan (1982), have used intrusive instrumentation. This fact casts a heavy shadow of doubt on the reliability of the measurements in the proximity of the wall. This is strongly indicated by the existing discrepancies between experimental data from different investigations at virtually identical "bulk" nominal conditions (Liu and Bankoff, 1990). Another plausible explanation for such discrepancies is the possible dependence of the local flow characteristics on bubble size. The consensus in most recent theoretical (Lance and Bataille, 1991) and experimental works (Liu and Bankoff, 1990) seems to be that the bubble size is a significant parameter. Of course, bubble size depends to a great extent on flow conditions.

Following the work of Thomas (1981), one can bracket the stable bubble size in turbulent flow within a range defined by a maximum size that can be stable against breakup (Hinze, 1955) and a minimum size that can resist coalescence (Thomas, 1981). The relevant mechanisms dictating both limits are related to turbulence characteristics of the continuous phase. As pointed out by Thomas (1981), bubbles with sizes within these limits are too large to coalesce and too small to be broken up; since they are all stable, the actual bubble distribution in the flow is likely to be determined by initial conditions, that is, injection conditions and experimental facility configuration. Despite the significant contributions of the existing experimental studies, the effects of bubble size and size distribution, possible effects of the injection method and conditions, and the possible dependence of the results on the design characteristics of the experimental facilities used remain unknown. In spite of these limitations, all of the available experimental data, obtained in low void fraction (<20%), dispersed, bubbly pipe flow, indicate the undisputed qualitative trend that the void fraction profile peaks close to the wall up to moderately high Reynolds numbers or at the center of the pipe for very high Reynolds numbers. Although the method of this study can be applied to both cases of void fraction peaking, the emphasis here has been placed on the wall peaking case.

In this work, a model is developed for steady-state two-phase, vertical, upward gas-liquid pipe flow in the bubbly regime, based on a set of simple turbulent flow equations. The two-phase mixture is considered as a single fluid of variable time-averaged local density whose functional form is prescribed. The radial variation of the density is a result of the nonuniform distribution of gas bubbles across the pipe. Wang et al. (1987) indicated that a turbulence-induced mechanism contributes to this phenomenon, while the Magnus effect or purely kinematical effects could also play an equally significant role. The void fraction distribution is axisymmetric for vertical flow and to a good approximation for horizontal flow with high bubble Froude numbers and consequently negligible stratification. The turbulent shear stress is created by the contribution of both velocity and density fluctuations, and it is modeled by utilizing a modified version of Prandtl's mixing length hypothesis. An expression for the shear stress is derived from a Reynolds-type equation which finally yields the velocity profile. The velocity profile can be calculated in conjunction with the density profile for given liquid and gas superficial velocities and average void fraction, taking into account the drift velocity (Zuber and Findlay, 1965). The friction multiplier can then be found from the velocity profile.

This model is phenomenological and bears all the restrictions of a single-fluid approach and a zero-order turbulence model. It is capable of providing good results for time-averaged flow quantities, but cannot accurately describe local turbulent aspects of the flow because interactive mechanisms through the interfaces are inherently absent. Furthermore, due to the choice of the radial void fraction distribution which accounts for the wall-peaking phenomenon, this analysis can render reasonable results only for bubbly pipe flows with Reynolds numbers in the range of validity of the assumed density profile (for moderate gas and liquid velocities). The general approach of the model, however, may be used for other regimes, such as annular or mist flows, by choosing suitable density distributions. Two-fluid models (Drew and Lahey, 1982; Beyerlein et al., 1985; Wang et al., 1987) which predict void fraction and velocity distributions from first principles certainly have a firmer physical basis; however, they are more complex, and require a better understanding of phase interaction and turbulence modification mechanisms than is currently available and, therefore, they necessitate the same, if not more, empirical information as the model used here. Thus, it is fair to say that the simplicity of this model and its methodology make it very attractive for engineering calculations.

Formulation of the Problem

We examine here adiabatic gas-liquid flows of axisymmetrically dispersed bubbles in a pipe of circular cross section. Such flows are commonly encountered in two-phase systems (such as petroleum, chemical, and geothermal), especially in cases where the orientation of the pipe is vertical and flow stratification is absent. Our objective is to develop a model of the flow, which will allow us to predict the friction multiplier necessary to calculate the pressure drop for engineering applications. The analysis has been carried out based on a number of assumptions:

- The time-averaged density variations of the fluid yield axisymmetric density profiles. This is a good approximation for flow in vertical channels and horizontal flows with small bubbles or high velocities. In the latter case, a high Froude number, based on bubble diameter, is characteristic. Naturally, this assumption does not apply to stratified flows which are characterized by low Froude numbers.
- The flow problem is treated as steady and two-dimensional, assuming that the longitudinal gradients are much smaller than the transverse ones. Thus, except for the pressure, all other derivatives in the axial direction are neglected.
- The flow is turbulent and steady after the appropriate time-averaging.
- Surface-tension effects in the momentum balance are neglected. This assumption is plausible for the flow under consideration, which is in accordance with the available experiments (Serizawa et al., 1975; Nakoryakov et al., 1981; Wang et al., 1987; Liu and Bankoff, 1990) in large size pipes with average bubble sizes greater than 1 mm.

The flow is governed by the Reynolds equations which are obtained in the usual manner. According to the previously stated assumptions, the governing equations reduce to:

$$\frac{dp}{dz}=\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{du}{dr}\right)+\frac{1}{r}\frac{d}{dr}(r\tau')-g\rho, \tag{1}$$

in which ρ, u and P are the local density, longitudinal velocity and pressure of the fluid, and τ' is the Reynolds stress. Density and velocity are functions of the radial coordinate alone, while pressure is a function only of the streamwise coordinate.

The Reynolds stress for a two-dimensional compressible flowfield (Schlichting, 1978) has the form:

$$\tau'=-(\rho\overline{u'v'}+u\overline{\rho'v'}+\overline{u'\rho'v'}+v\overline{\rho'u'}) \tag{2}$$

where u and v are time-averaged components of the velocity vector in the streamwise and radial directions, respectively, and ρ is the time-averaged density; the primes denote turbulent fluctuations, and the overbar the time average. The overbar has been dropped for simplicity in the case where products of turbulent quantities are not involved. In the case of one-dimensional flow, the average radial velocity is very small and can be assumed to be zero; therefore, the term $v(\overline{\rho'u'})$ vanishes. Since density variations in the flow can be effected primarily through the action of velocity fluctuations, the relative turbulence intensity u'/u can be thought of as the upperbound for ρ'/ρ (as argued by Clark and Flemmer, 1985). Therefore, considering that u'/u, $\rho'/\rho \ll 1$, the triple correlation term may be neglected in comparison with these two [since $(\rho'u'v')/(\rho u'v')\cong\rho'/\rho$ and $(\rho'u'v')/(u\rho'v')\cong u'/u$]. Subsequently, the only contribution to the turbulent stress comes from the momentum exchange terms stemming from velocity fluctuations and from the combined effect of density and velocity fluctuations. Thus, we may write

$$\tau'=-(\rho\overline{u'v'}+u\overline{\rho'v'}). \tag{3}$$

The usual eddy diffusivity term has two components, the second being due to density fluctuations. This second term always increases the shear stress in a two-phase mixture. It should be noted here that Clark and Flemmer (1985) have included a third component in the velocity decomposition representing the perturbation created by the wakes of fast-moving bubbles. This third component introduced additional Reynolds-type stresses which Clark and Flemmer (1985) have modeled. This additional perturbation velocity owes its existence to the buoyancy forces that are responsible for the faster motion of the bubbles compared to the surrounding fluid. Since a buoyancy force has been taken into account explicitly in the equation of motion (last term of Eq. 1) and our approach involves a variable-density, single fluid, such an additional velocity perturbation is not necessary. Equation 1 can be easily integrated since the pressure gradient is independent of the radial coordinate. Therefore, substituting for the turbulent stress from Eq. 3, we obtain the following force balance equation:

$$\frac{r}{2}\frac{dp}{dz}=\mu\frac{du}{dr}-\rho\overline{u'v'}-u\overline{\rho'v'}-\frac{g}{r}\int_0^r\rho r dr. \tag{4}$$

The net shear stress experienced by the fluid at any radial position in the pipe is the sum of the viscous and turbulent components:

$$\tau=\mu\frac{du}{dr}-\rho\overline{u'v'}-u\overline{\rho'v'}, \tag{5}$$

Since the origin of the selected coordinate system is the centerline of the pipe, du/dr, as well as τ, is negative. All of the equations are consistent with this convention. Substituting Eq. 5 into Eq. 4 we obtain

$$\tau=\frac{r}{2}\frac{dP}{dz}+\frac{g}{r}\int_0^r\rho r dr. \tag{6}$$

This equation yields the wall shear stress:

$$\tau_w=\frac{R}{2}\frac{dP}{dz}+\frac{g}{R}\int_0^R\rho r dr. \tag{7}$$

Eliminating the pressure gradient dP/dz between the above equations, we obtain the following expression for the shear stress:

$$\tau=\tau_w\frac{r}{R}+g\left[\frac{1}{r}\int_0^r\rho r dr-\frac{r}{R^2}\int_0^R\rho r dr\right]. \tag{8}$$

The wall shear stress equals the viscous stress in the laminar sublayer, where the turbulent stress is absent. Thus,

$$\tau_w = \mu \left. \frac{du}{dr} \right|_{r=R}. \tag{9}$$

It is necessary at this point to introduce a closure for the turbulent stress τ' to solve the equation. In single-phase flow, a variety of hypotheses can be found in the literature that would yield a suitable form for a closure equation. Prandtl's mixing length theory (1925), Taylor's vorticity hypothesis (1932), and von Karman's similarity hypothesis (1930) have led to similar results for the velocity distribution and successfully predict the frictional pressure drop for single-phase pipe flow. Here, a modified version of Prandtl's mixing length hypothesis will be used because of its simplicity:

$$-\overline{v'u'} = l_u^2 \frac{du}{dr} \left| \frac{du}{dr} \right| \tag{10}$$

and in analogy to Eq. 10,

$$-\overline{\rho' v'} = l_u l_\rho \frac{du}{dr} \left| \frac{d\rho}{dr} \right|, \tag{11}$$

where l_u, l_ρ are the two mixing length scales. The absolute values in Eqs. 10 and 11 are necessary to satisfy the dissipation inequality dictated by the second law of thermodynamics,

$$\tau' \frac{du}{dr} \geq 0, \tag{12}$$

which states that the Reynolds stress must have the same sign as the velocity gradient.

According to the above closure conditions and Eq. 5, the shear stress can be written as

$$\tau = \mu \frac{du}{dr} + \rho l_u^2 \frac{du}{dr} \left| \frac{du}{dr} \right| + u l_u l_\rho \frac{du}{dr} \left| \frac{d\rho}{dr} \right|. \tag{13}$$

It must be pointed out that the closure equation predicts zero shear whenever du/dr is zero. Although this is correct whenever the velocity peaks at the center of the pipe, it is not correct if du/dr is zero off-center. This is one of the general shortcomings of zero-order turbulence closure equations, such as the mixing length hypothesis. Despite this physical inadequacy of the mixing length hypothesis, it has been used in pipe flows, because of its simplicity, with reasonable success. In the case of pipe flows with bubbles, one is tempted to use the same hypothesis, which, despite the inaccurate representation of shear stress, may yield satisfactory results for the integral quantities.

Thus, Eq. 8 becomes:

$$|\tau_w| \frac{r}{R} = -\mu \frac{du}{dr} - \rho l_u^2 \frac{du}{dr} \left| \frac{du}{dr} \right| - u l_u l_\rho \frac{du}{dr} \left| \frac{d\rho}{dr} \right| + g \left[\frac{1}{r} \int_0^r \rho r dr - \frac{r}{R^2} \int_0^R \rho r dr \right], \tag{14}$$

where $|\tau_w|$ is the magnitude of the negative wall shear stress. The viscous part of the shear stress will be based on the two-phase viscosity, which is a function of the void fraction, as introduced by Ishii (1977) and successfully used by Ishii and Zuber (1979).

Equation 14 is made nondimensional by introducing the following variables:

$$\rho^+ = \frac{\rho}{\rho_L}, \quad \mu^+ = \frac{\mu}{\mu_L}, \quad l_u^+ = \frac{l_u}{R}, \quad l_\rho^+ = \frac{l_\rho}{R},$$
$$x = \frac{r}{R}, \quad u^+ = \frac{u}{V^*} = \frac{u}{\sqrt{|\tau_w|/\rho_L}}. \tag{14a}$$

The final dimensionless form of the momentum equation then becomes:

$$\rho^+ l_u^{+2} \frac{du^+}{dx} \left| \frac{du^+}{dx} \right| + \left\{ \frac{\mu^+}{Re^*} + u^+ l_u^+ l_\rho^+ \left| \frac{d\rho^+}{dx} \right| \right\} \frac{du^+}{dx} + \left\{ x + \frac{1}{(Fr^*)^2} \left[xG(1) - \frac{1}{x} G(x) \right] \right\} = 0 \tag{15}$$

where

$$G(x) = \int_0^x \rho^+ x dx. \tag{16}$$

The Reynolds number,

$$Re^* = \frac{\rho_L V^* R}{\mu_L}, \tag{17}$$

is based on the shear velocity, V^*, and on the properties of the liquid, while a Froude number is also defined in terms of the same velocity and the pipe radius:

$$Fr^* = \frac{V^*}{\sqrt{gR}}. \tag{18}$$

Mixing Length and Radial Density Distributions

To solve the problem, we need to provide an expression for the two length scales l_u^+ and l_ρ^+, as well as for the radial density distribution. Regarding the velocity mixing length l_u^+, it appears that a good approximation is to take the empirical relationship developed by Nikuradse (1932) for pipe flows:

$$l^+ = l_u^+ = 0.14 - 0.08x^2 - 0.06x^4. \tag{19}$$

The density mixing length l_ρ^+ is not very easily determined or approximated. At first, it appears that there are no experimental data from which l_ρ^+ can be directly determined. Second, it appears that the size of bubbles and the void fraction at a given location should both play a role in the closure equation for l_ρ^+; similarly, the size distribution of bubbles, bubble shape and interbubble distance should come into this equation as well. Third, even if one lists all these variables in an expression for l_ρ^+, the mechanism by which l_ρ^+ is affected by all of these is not determined (one has to answer such questions as: "How does the mixing length compare to the bubble size?"; "Is the influence of the bubble size on the mixing length uniform, that is, does it depend on Re, position or concentration of the bubbles?"; "Can the bubbles absorb momentum from eddies and how much?"). In the case of small-particle flows (Michaelides, 1984, 1986), it was observed that reliable results could be obtained by assuming $l_u^+ = l_\rho^+$. This approximation is certainly acceptable for small bubbles. In most of the available experimental investigations, however, bubbles can be of the same order as the mixing length. One can argue that for larger, deformable bubbles the length scales of the deformations are smaller than l_u and thus the above approximation may be valid. Taking into account that the lack of relevant information does not allow a better approximation to be made for l_ρ^+, we have assumed that

$$l^+ = l_u^+ = l_\rho^+. \tag{20}$$

Comparison of the model predictions with experiments will indirectly put this assumption to the test; however, cancellation of error due to the other assumptions is always possible.

The radial density distribution depends on the radial void fraction distribution. Local void fraction measurements indicate that in the bubble flow regime the local void fraction exhibits a peak near the wall in the range of Reynolds numbers up to at least 80,000 (Nakoryakov et al., 1981). To take this phenomenon into account we have chosen a local void fraction profile of the following form:

$$\frac{\alpha - \alpha_m}{\alpha_c - \alpha_m} = \left[1 - \left(\frac{x}{x_m}\right)^m\right]^n. \tag{21}$$

In this expression α_c is the local void fraction at the centerline of the pipe, and α_m is the maximum value of the void fraction at the radial location $x = x_m$. This peaking location measured from the wall can be considered to be of the order of the average bubble radius throughout the range of cross-sectional averaged void fractions corresponding to bubbly, dispersed flow (average void fraction less than 20%). Liu and Bankoff (1990) have provided detailed estimates of average bubble size (from bubble chord length distributions). A careful examination of their data shows that our approximation is certainly good to within the experimental error, introduced by the intrusive void-fraction measuring techniques near the wall and by the deduction of bubble size from chord length distributions. Bubble size estimates by Serizawa et al. (1975), Nakoryakov et al. (1981), and Wang et al. (1987), although they did not carry out explicit bubble size measurements, also support this assumption. Thus, we have taken $x_m = 1.0 - \langle d_b \rangle / (2R)$, where $\langle d_b \rangle$ is the average bubble size.

All space-averaged quantities are defined hereafter as:

$$\langle q \rangle = \frac{\int_A q\,dA}{\int_A dA}$$

The flow is assumed to be adiabatic, and thus the local void fraction at the wall $\alpha(1)$ is taken to equal zero (Zuber and Findlay, 1965) because bubbles cannot attach to the wall without the presence of a liquid layer. This condition relates the quantity α_m to α_c in a very definite way, for given n and m:

$$\alpha_c = \alpha_m \left[1 - \left(1 - \frac{1}{x_m^m}\right)^{-n}\right]. \tag{22}$$

Using Eq. 22 we can relate α directly to the cross-sectional averaged void fraction:

$$\alpha = \langle \alpha \rangle \frac{\sum_{k=1}^{n} (-)^k \binom{n}{k} (1 - x^{mk}) x_m^{-mk}}{\sum_{k=1}^{n} (-)^k \binom{n}{k} \frac{mk}{mk+2} x_m^{-mk}} \tag{23}$$

for given n and m.

Consequently, the two exponents m and n must be specified *a priori* to generate a void fraction profile with a prescribed $\langle \alpha \rangle$ (which is usually a characteristic flow quantity when presenting experimental data). The constraint imposed on the choice of n and m is that α_m, α_c, and $\langle \alpha \rangle$ must all be positive and less than unity. The exponent n for our case of dispersed, bubbly flow within the regime that exhibits a void fraction maximum near the wall must be even. The void fraction profiles measured by Serizawa et al. (1975), Nakoryakov et al. (1981), Wang et al. (1987), and Liu and Bankoff (1990) all display a rather sharp peak; values of $n > 2$ give maxima that are too flat, thus $n = 2$ is the only acceptable choice and will be used in all of our future calculations. It must be emphasized that Eq. 21 is a general expression for the void fraction profile and that n and x_m may be chosen so that the void fraction exhibits a peak at any desired location in the pipe. For example, if the peak occurred at the center of the pipe, the choice would be $x_m = 1$, $n = 1$, and the void fraction profile family would then be identical to the one proposed by Bankoff (1960) and Zuber and Findlay (1965). In such a case, α_m would be identically zero because of the condition $\alpha(1) = 0$. The exponent m is not fixed and will be calculated in conjunction with the velocity profile through an iterative procedure that will be described later.

The local void fraction profile expressed by Eq. 21 will yield a radial density distribution of similar form:

$$\frac{\rho^+ - \rho_m^+}{\rho_c^+ - \rho_m^+} = \left[1 - \left(\frac{x}{x_m}\right)^m\right]^n, \tag{24}$$

given that, in general, the dimensionless local density of the mixture is:

$$\rho^+ = 1 + \left(\frac{\rho_G}{\rho_L} - 1\right)\alpha. \tag{25}$$

The cross-sectional average density of the mixture is then given as:

$$\langle \rho \rangle = \frac{\int_A \rho\,dA}{\int_A dA} = 2\rho_L G(1). \tag{26}$$

It must be noted that the choice of density profile parameters has been made for the case of up-flow only. Wall peaking of the void fraction has been observed in down-flow situations (Wang et al., 1987) as well, but the peaks are much less pronounced than in up-flow cases. A different choice of parameters may be required for the down-flow case which is not addressed in this study.

Velocity Profile and Solution Procedure

Equation 15 can be easily solved for the velocity gradient provided that the necessary attention is paid to determining its correct sign, and given that its absolute value is involved in the equation:

$$\frac{du^+}{dx}=\frac{C}{|C|}\left[\frac{B-\sqrt{B^2+4A|C|}}{2A}\right], \tag{27}$$

where

$$A=\rho^+ l_u^{+2}, \tag{28a}$$

$$B=u^+ l_u^+ l_\rho^+ \left|\frac{d\rho^+}{dx}\right|+\frac{\mu^+}{Re^*} \tag{28b}$$

and

$$\begin{aligned}C&=x+\frac{1}{(Fr^*)^2}\left(xG(1)-\frac{1}{x}G(x)\right)\\&=x\left[1+\frac{(\rho_c^+-\rho_m^+)}{(Fr^*)^2}\sum_{k=1}^{n}\binom{n}{k}\frac{(-)^k}{mk+2}\frac{(1-x^{mk})}{x_m^{mk}}\right].\end{aligned} \tag{28c}$$

The quantities A and B are always positive; therefore, the sign of the velocity gradient depends only on the sign of C (Eq. 27). Actually it can be easily shown that sign$(du^+/dx) = -$sign$(C) = -C/|C|$. Quantity C can be positive or negative depending only on the density distribution and the Froude number. In the case of horizontal flow, $C=x$ and is always positive; consequently, the velocity gradient will be negative (for the chosen coordinate system) over the full range of the radial coordinate. For vertical flow, quantity C and therefore the velocity gradient may locally change sign due to the gravitational effect and the phase distribution.

Equation 27 is a first-order nonlinear differential equation subject to the boundary condition $u^+=0$ at $x=1$ (no slip at the wall of the pipe). A classical viscous sublayer ($0<u^+<5$) has been assumed near the wall. Within this sublayer the velocity profile is determined by integrating $(du^+/dx) = -(Re^*/\mu^+)$ until $u^+=5$. Beyond this point and outside the viscous sublayer, the velocity profile is obtained by integrating Eq. 27, which satisfies the condition $du^+/dx=0$ at the pipe centerline since C is identically zero at that location. No adjustable parameters have been used for this integration other than the constants given in the closure equations.

Given the velocity profile, the average volumetric flux densities ("superficial" velocities) of the gas, V_G^s, and the liquid, V_L^s, can be calculated provided that the effect of the drift velocity, V_{GL}, (Zuber and Findlay, 1965) is taken into account. Thus,

$$V_G^s=\langle\alpha u\rangle+\langle\alpha V_{GL}\rangle, \tag{29a}$$

$$V_L^s=\langle u\rangle-V_G^s \tag{29b}$$

with

$$\langle u\rangle=V^*2\int_0^1 u^+x\,dx. \tag{30}$$

For the drift velocity, we have adopted the expression used by Ishii and Zuber (1979):

$$V_{GL}=V_{b\infty}(1-\alpha)^{3/2}g(\alpha,d_b^*) \tag{31}$$

where d_b^* is a dimensionless bubble diameter, $V_{b\infty}$ is the bubble rise velocity in an infinite medium, and $g(\alpha, d_b^*)$ is a known function of the void fraction and of the dimensionless bubble diameter. The definitions of these quantities in the Appendix are those used by Ishii and Zuber (1979). The dimensionless bubble diameter, d_b^*, in our case was based on the average bubble size, $\langle d_b\rangle$, and thus was assumed constant in the radial direction. This is a reasonable assumption considering that in the experiments of Liu and Bankoff (1990) the bubble size does not vary considerably in the radial direction. Most of that variation could be attributed to the deformation of the bubbles near the wall because of shear and/or to the fact that the direct quantity measured was the chord length.

One of the main objectives of this article is to determine the frictional pressure drop and the friction multiplier for the two-phase flow under consideration. The wall shear stress can be calculated from $\tau_w=\rho_L V^{*2}$ and the two-phase, frictional pressure drop can then be determined as:

$$\left[\frac{dp}{dz}\right]_{TPf}=\frac{2}{R}\tau_W. \tag{32}$$

It is customary in the two-phase flow literature to present the frictional pressure drop in terms of the friction multiplier ϕ_{LO} defined as:

$$\phi_{LO}^2=\frac{[dP/dz]_{TPf}}{[dP/dz]_{LOf}}, \tag{33}$$

where

$$\left[\frac{dP}{dz}\right]_{LOf}=f_{LO}\frac{(\rho_L V_L^s+\rho_G V_G^s)^2}{2D\rho_L} \tag{34}$$

is the frictional pressure drop for a single-phase liquid flow at the same mass flux and in the same geometry, and f_{LO} is the usual single-phase friction factor. Thus, the friction multiplier can be evaluated in terms of quantities that will be obtained from the model:

$$\phi_{LO}^2=\frac{8}{f_{LO}}\left(\frac{\rho_L V_L^s+\rho_G V_G^s}{\rho_L V^*}\right)^{-2}. \tag{35}$$

In practice, the average volumetric flux densities are usually known so the objective is to obtain a solution corresponding to known V_L^s and V_G^s, the properties of the fluids involved, and the geometry of the pipe. This model requires additional information, such as the average void fraction $\langle \alpha \rangle$ and the average bubble size $\langle d_b \rangle$. So, given V_L^s, V_G^s, $\langle \alpha \rangle$, $\langle d_b \rangle$, properties of the fluids and the pipe geometry, we use Eqs. 29a and 29b to calculate the exponent m and the friction velocity V^* using an iterative scheme. To this effect, Eq. 27 is integrated at every step for the velocity profile using the void fraction profile family (Eq. 29). The friction multiplier is then calculated from Eq. 35. Regarding the necessity of knowing $\langle \alpha \rangle$ and $\langle d_b \rangle$, the experimental evidence shows that the relationship between the average void fraction and the average volumetric flux densities is not unique and strongly indicates (particularly in the study of Liu and Bankoff, 1990) that the distributions of the flow variables depend on the bubble size. It has not been established, however, whether the bubble size is a function of the flow conditions in the pipe alone or is determined to a certain extent by the injection method and the geometry of the injection system, as argued by Thomas (1981). To obtain predictions for the friction multiplier using this model when only the average volumetric flux densities are known, a correlation relating the average void fraction to these quantities (for example, Rouhani, 1969) will have to be used, and an estimate of the average bubble size must be known. For engineering calculations, it is possible to estimate an average bubble size by using the results from the work of Thomas (1981) which has been discussed previously. The average bubble size can be approximated as the average of the maximum size that is stable against breakup and the minimum size that is stable against coalescence. Such an approximation can be reasonable if the mechanism that introduces bubbles in the flow generates a wide spectrum of sizes. For example, such was the case in Nakoryakov et al. (1981) where a porous plug was used for bubble generation. This approximation may not be exactly valid in cases such as that of Liu and Bankoff (1990) where individual needle injectors were used and the bubble size was more likely tied to the injector characteristics and the flow around these injectors.

Results and Comparison with Experiments

In comparing our results to experiments we have used the measured average void fraction and measured or estimated average bubble size to obtain the void and liquid velocity distributions as well as the friction multiplier. The study of Liu and Bankoff (1990) is the only one providing bubble chord-length distribution data from which an average bubble size can be deduced. Our analysis of Liu and Bankoff's bubble size data shows that the average bubble size scaled by the pipe size correlates very well to the volumetric gas-flow ratio $\langle \beta \rangle = V_G^s / (V_L^s + V_G^s)$. The correlation was $\langle d_b \rangle / R \simeq (0.192 \langle \beta \rangle + 0.129)$ with a correlation coefficient of 99%. This certainly is not a universal relationship, and we have used it for convenience in calculating the void and liquid velocity distributions to compare our results with the experimental ones of Liu and Bankoff (1990). The above relation was also used to estimate the bubble size for the experiments of Wang et al. (1987) because, on one hand, the method of injection used in their study was effectively the same as that of Liu and Bankoff (1990) and, on the other, the reported bubble size range from their visual observations (Wang, 1985) also matched that of Liu and Bankoff (1990)

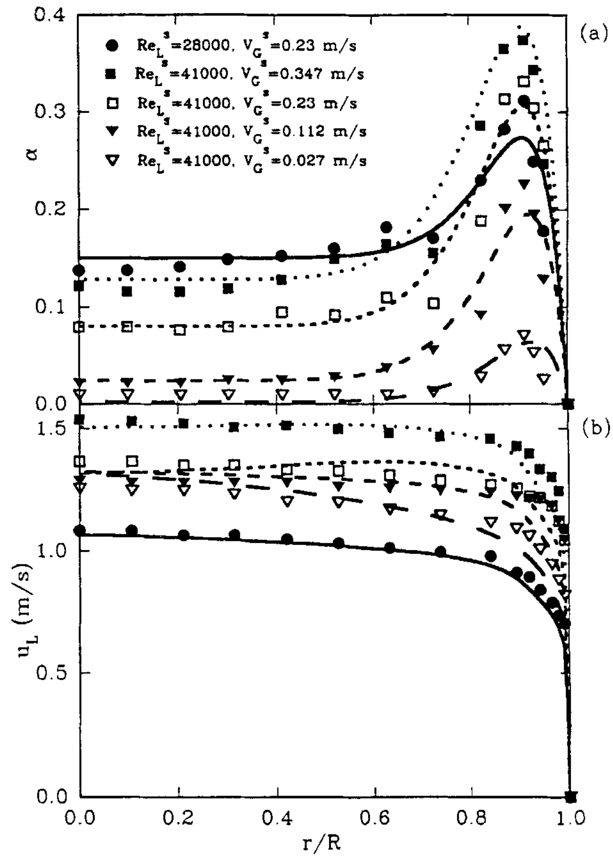

Figure 1. Comparison of void fraction (a) and liquid velocity (b) distributions in vertical-upward pipe flow, calculated using our model (lines) with experimental data of Liu and Bankoff (1990).

satisfactorily. For our comparisons of the predicted friction multipliers with the experimental data of Nakoryakov et al. (1981), who made extensive measurements of the wall shear stress using an electrochemical method, we used different bubble sizes within the reported range of 2-4 mm for the dispersed bubble regime. Estimates of average bubble size based on Thomas (1981) for the various flow conditions of these experiments also fall within the experimentally observed range.

Results of our model for bubbly air-water flow corresponding to various conditions in the experiments of Liu and Bankoff (1990) are presented in Figure 1. Both the void fraction distribution and the liquid velocity distribution in the radial coordinate are reasonably well represented by the model. The local liquid velocity was calculated from:

$$u_L(x) = \{1 - \alpha(x)\} u(x) - \alpha(x) V_{GL}(x). \tag{36}$$

Figure 1a shows that the assumption of the peak void fraction location (one bubble radius from the wall) is quite reasonable since the calculated maxima are in fairly good agreement (within the experimental uncertainty) with the measured ones. Most of the deviations between local measured and calculated values are also within experimental uncertainty. It should, however, be noted that at high gas flow rates the experimental data exhibit a very weak second peak of the void fraction occurring before the main peak. This is not represented by the void fraction profile family used in this model. The liquid velocity profiles shown in Figure 1b correspond to the void distributions in Figure 1a. The calculated velocity profiles generally agree well with the experimental ones. Most of the differences occur near the wall where the measured velocities are usually higher. Test measurements of Liu and Bankoff (1990) in single-phase flow (water only) using their facility and methods were generally in good agreement with the data of Laufer (1954) and that the only discrepancy appeared near the wall where the average liquid velocities of Liu and Bankoff were larger than those of Laufer (1954).

The most important difference between the experimental data of Liu and Bankoff and the results of this model is a qualitative one. In several of the examined cases, the calculations show a local maximum of the liquid velocity off the center of the pipe and closer to the wall, while the measured velocity profiles do not. This off-center peak of the liquid velocity has been observed in the experiments of Malnes (1966), Nakoryakov et al. (1981), Theofanous and Sullivan (1982), and Wang et al. (1987). The study of Serizawa et al. (1975) is the only one, other than that of Liu and Bankoff (1990), that does not display this off-center peak. It is evident that the majority of experimental measurements support the existence of the off-center liquid velocity peak, and our model predicts it on the basis of the buoyancy of the dispersed phase which can carry the slower liquid closer to the wall at higher velocities. This mechanism is to a great extent responsible for the increase in the momentum of the fluid near the wall which is displayed by all the experiments and predicted by the model. The fact that the off-center peaking of the liquid velocity does not appear in the experimental results of Liu and Bankoff (1990) but is predicted by the model, apart from possible experimental error, could be due to a number of mechanisms not taken into account by this model, such as effects of the injection conditions (that may play a role in determining the "fully developed" flow as Liu and Bankoff argued), bubble-size distribution and bubble-size effect on the turbulent quantities. Comparisons of the predictions of this model with the data of Serizawa et al. (1975) have also been carried out with similar results and are therefore not presented here.

To test the model further our data were compared to those by Wang et al. (1987), as summarized in Figure 2. Again the calculated void profiles (Figures 2a and 2c) are fairly well represented, particularly for cases corresponding to a lower "superficial" Reynolds number, $Re_L^s = (V_L^s \rho_L D)/\mu_L$ (Figure 2a). The calculated velocity profiles (Figures 2b and 2d) are also in reasonable agreement with the experimental ones. Both measured and calculated profiles exhibit the off-center velocity peaking, the latter generally overpredicting the peak velocity. This is not surprising given the simple turbulence model used which cannot accurately describe the local turbulent characteristics of the flow. Comparisons with the data of Nakoryakov et al. (1981) also showed equally satisfactory results.

Nakoryakov et al. (1981) have also carried out detailed measurements of wall shear stress using an electrochemical method. Results for the friction multiplier from this model and existing correlations are compared with the results from their experiments for four different "superficial" Reynolds numbers (Figure 3). The friction multiplier is presented as a function of the volumetric gas-flow ratio $\langle\beta\rangle$. The experimental data show a dramatic increase in the friction multiplier within the dispersed

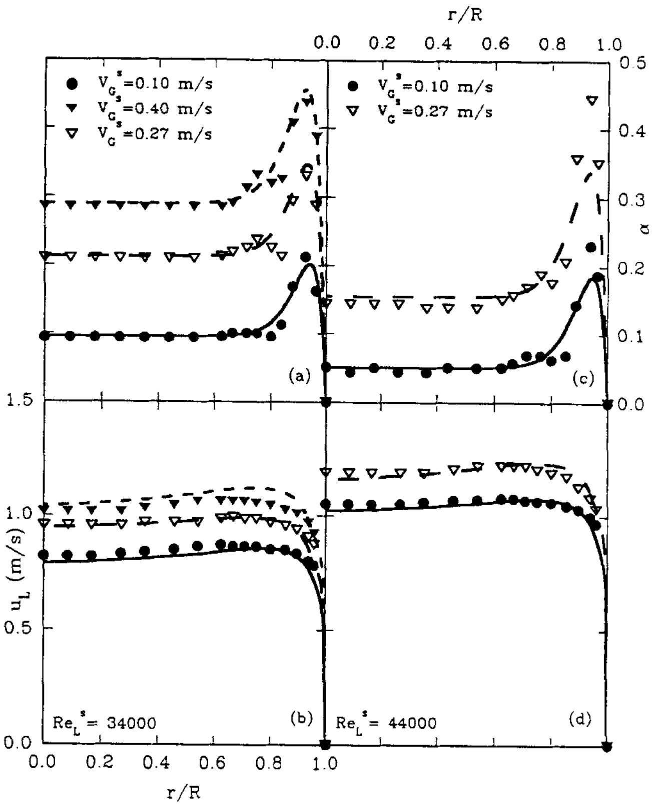

Figure 2. Comparison of void fraction (a, c) and liquid velocity (b, d) distributions in vertical-upward pipe flow, calculated using our model (lines) with experimental data of Wang et al. (1987).

bubble regime when the gas phase was introduced. The friction multiplier then levels off to a high value as $\langle\beta\rangle$ increases into the churn-bubbly regime and later drops off drastically as the flow goes into the slug regime. This latter portion of the data is not included in Figure 3 since it is completely outside the range of validity of the model. It is evident from Figure 3 that all the correlations examined here (Lockhart and Martinelli, 1949; Lottes and Flinn, 1956; Dukler et al., 1964; Chisholm, 1973), which is based on the earlier graphic correlation of Baroczy (1965) and Friedel (1979); and the homogeneous one underpredict the friction multiplier by a very large factor. The correlation of Armand (1950), which renders results of the same magnitude as the other correlations examined here, has not been included because its results have already been compared to the measurements by Nakoryakov et al. (1981).

The more recent correlation of Beattie and Whalley (1982) has also not been included in this comparison because it covers a wide range of bubble and annular flow regimes and, as pointed out by the authors, it shows a poorer performance in the bubble regime. Three results of our model are shown in Figure 3, each one corresponding to a different bubble size (2, 3 and 4 mm) within the range reported by Nakoryakov et al. (1981) which was a result of their visual observations and not of detailed measurements. Unlike the correlations, the friction multipliers predicted by the model are representative of the data in magnitude. The quantitative agreement seems to be better for moderately high Reynolds numbers (Figures 3c and 3d). In terms of trends they too display a considerable increase

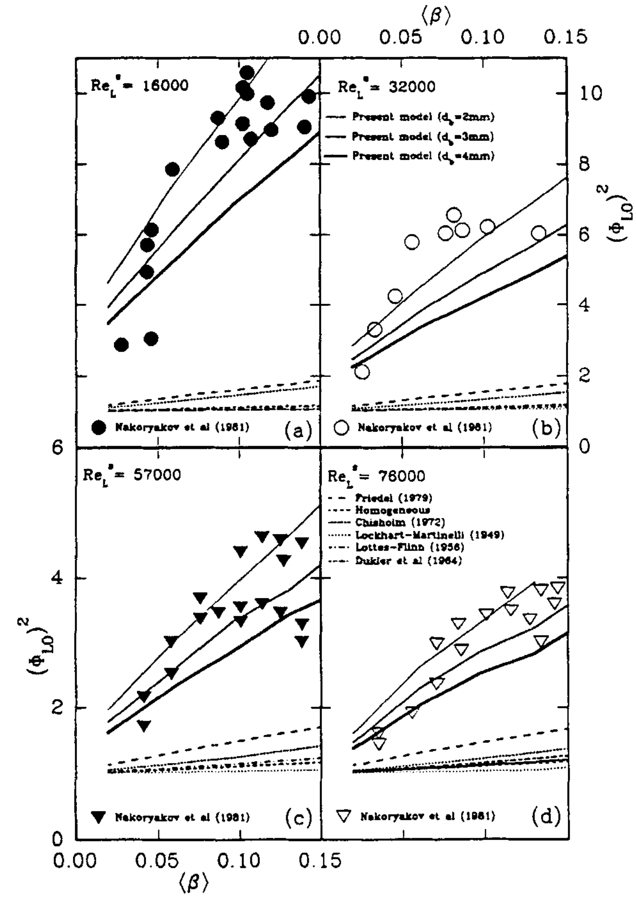

Figure 3. Friction multiplier obtained using our model vs. experimental results of Nakoryakov et al. (1981) and six representative correlations, for various "superficial" Reynolds numbers.

upon introduction of the dispersed phase (increasing $\langle\beta\rangle$ or, equivalently, $\langle\alpha\rangle$), but cannot predict the plateau of the friction multiplier displayed by the data in the bubble-churn regime where the near wall-peaking of the void fraction disappears. The correlations are also unable to predict this plateau. The decrease of the friction multiplier with increasing liquid "superficial" velocity (or Reynolds number) is also well represented by the results of the model.

The present model can reasonably predict the strong increase in the frictional pressure drop with increasing gas content, despite the use of a simple turbulence model that is not expected to represent very well the local turbulent quantities. This indicates that, in the dispersed bubble regime, the increase of the frictional pressure drop is largely due to the strong buoyancy effect introduced by the presence of bubbles moving along the wall and the subsequent near-the-wall peak of the void fraction. This is further supported by the following facts: (a) in the dispersed bubble regime, the considerable increase of the friction multiplier with increasing gas flow rate at a constant liquid flow rate (Nakoryakov et al., 1981) coincides with the existence of sharper near-the-wall void fraction peaks; (b) this increase is curbed as the near-the-wall void fraction peak disappears in the bubbly-churn regime (Nakoryakov et al., 1981); (c) the friction multiplier sharply decreases as the gas flow rate is further increased and the flow goes into the slug regime where the void fraction is markedly low near the wall and its maximum is at the center of the pipe cross section (Nakoryakov et al., 1981); and (d) in the dispersed bubble regime, the decrease of the friction multiplier with increasing liquid flow rate coincides with the decrease of the off-the-wall void fraction peak (Nakoryakov et al., 1981; Wang et al., 1987; Liu and Bankoff, 1990). The existence and magnitude of the void fraction peaks, to which the observed variation of the friction multiplier seems to be related, may, of course, be influenced by the turbulence structure as suggested by various authors (Drew and Lahey, 1982; Wang et al., 1987). The present model does not include such a mechanism and does not explicitly calculate the void distribution.

The influence of bubble size in this model is limited to determine the near-the-wall void fraction peak location and the local drift velocity. According to the justification provided in a previous section, any direct influence of bubble size on the turbulent quantities has not been included. These facts and a smaller bubble size make the void fraction peak sharper and bring it closer to the wall. Figure 3 shows that the model indicates an increasing friction multiplier with decreasing bubble size. To confirm this as well as the influence of the bubble size on the turbulent aspects of the flow and the void distribution, it will be necessary to carry out local measurements similar to those of Liu and Bankoff (1990), including wall shear stress measurements and varying the bubble size. Such experiments are not yet available.

Conclusions

A phenomenological model was developed for steady-state axisymmetrically dispersed bubbly flows in pipes, based on a modified mixing length turbulence model and a void fraction profile family to represent the experimentally observed near-wall void fraction peaking. The gas-liquid mixture was considered as a homogeneous fluid of variable density across the pipe cross section. The eddy diffusivity of this two-phase system was enhanced by including another term due to the variation of the density. Thus, the Reynolds stresses for the flow exhibited two principal terms, which were calculated according to the mixing length hypothesis. The effect of the relative velocity between phases has been taken into account via the drift velocity, while the effect of bubble size was partially taken into account.

This model may be used to predict liquid velocity distributions and the frictional pressure drop. The calculated void fraction, liquid velocity profiles, and friction multipliers agree well with experimental data. The friction multipliers predicted by this model are a considerable improvement over those offered by existing correlations for low and moderately high "superficial" Reynolds numbers (up to roughly 80,000). Pressure drop correlations underpredict the friction multipliers by large factors (as high as 6) in this range of the dispersed bubble regime. It is important to point out that the model presented here is a mechanistic one and attempts to describe the average quantities of interest in dispersed bubbly pipe flows regardless of details such as distribution of bubble size and/or internal interactions. Also, a number of parameters in this study (such as the mixing lengths) were determined rather arbitrarily and along the lines of the corresponding single-phase flow, because of the lack of relevant experimental information. Nevertheless, the model describes the time-averaged flow and predicts flow quantities that are in good agreement with experimental data without involving adjustable constants. The simplicity and satisfactory performance of the model render it easy to use for engineering applications, despite its limitations primarily stemming from the use of its closure equations.

Notation

A = pipe cross-sectional area
d_b = bubble diameter
d_b^* = dimensionless bubble diameter (Ishii and Zuber, 1979)
D = pipe diameter
f_{LO} = liquid friction factor
Fr^* = Froude number based on shear velocity
g = acceleration of gravity
l_u, l_ρ = mixing length scales
m = first exponent of empirical void fraction profile
n = second exponent of empirical void fraction profile
P = pressure
r = radial coordinate
R = pipe radius
Re^* = Reynolds number based on shear velocity
Re_L^s = Reynolds number based on liquid "superficial" velocity
u = axial mean velocity
u' = axial velocity fluctuation
v = radial mean velocity
v' = radial velocity fluctuation
V^* = shear velocity
$V_{b\infty}$ = bubble rise velocity in an infinite medium
V_G^s = average volumetric flux density ("superficial" velocity) of the gas
V_L^s = average volumetric flux density ("superficial" velocity) of the liquid
V_{GL} = drift velocity
x = dimensionless radial coordinate
x_m = peak void fraction location
z = axial coordinate

Greek letters

α = void fraction
α_c = centerline void fraction
α_m = maximum void fraction
$\langle\beta\rangle$ = volumetric gas-flow ratio
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
ρ_c = centerline density
ρ_m = maximum density
σ = surface tension coefficient
τ' = Reynolds shear stress
τ_w = wall shear stress
ϕ_{LO}^2 = friction multiplier

Others

$(\ )_L$ = liquid
$(\ )_G$ = gas
$(\ )^+$ = dimensionless quantity
$(\bar{\ })$ = time average
$\langle\ \rangle$ = cross-sectional average

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Appendix

Drift-flux variables and functions as in Ishii and Zuber (1979):

$$V_{b\infty} = [4\sigma g(\rho_L - \rho_G)/\rho_L^2]^{0.25} \tag{A1}$$

$$d_b^* = \langle d_b \rangle \left[\frac{g}{\nu_L^2} \left(1 - \frac{\rho_G}{\rho_L}\right) \right]^{1/3} \tag{A2}$$

$$\mu^+ = \left(1 - \frac{\alpha}{\alpha_{\max}}\right)^{-2.5\alpha_{\max}(\mu_G + 0.4\mu_L)/(\mu_G + \mu_L)} \quad \text{with } \alpha_{\max} = 0.95 \tag{A3}$$

$$f(\alpha) = \frac{\sqrt{(1-\alpha)}}{\mu^+} \tag{A4}$$

$$\psi(d_b^*) = 0.55[(1 + 0.01 d_b^{*3})^{4/7}]^{3/4} \tag{A5}$$

$$g(\alpha, d_b^*) = f(\alpha)\, \frac{1 + \psi(d_b^*)}{1 + \psi(d_b^*) f(\alpha)^{6/7}} \tag{A6}$$

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